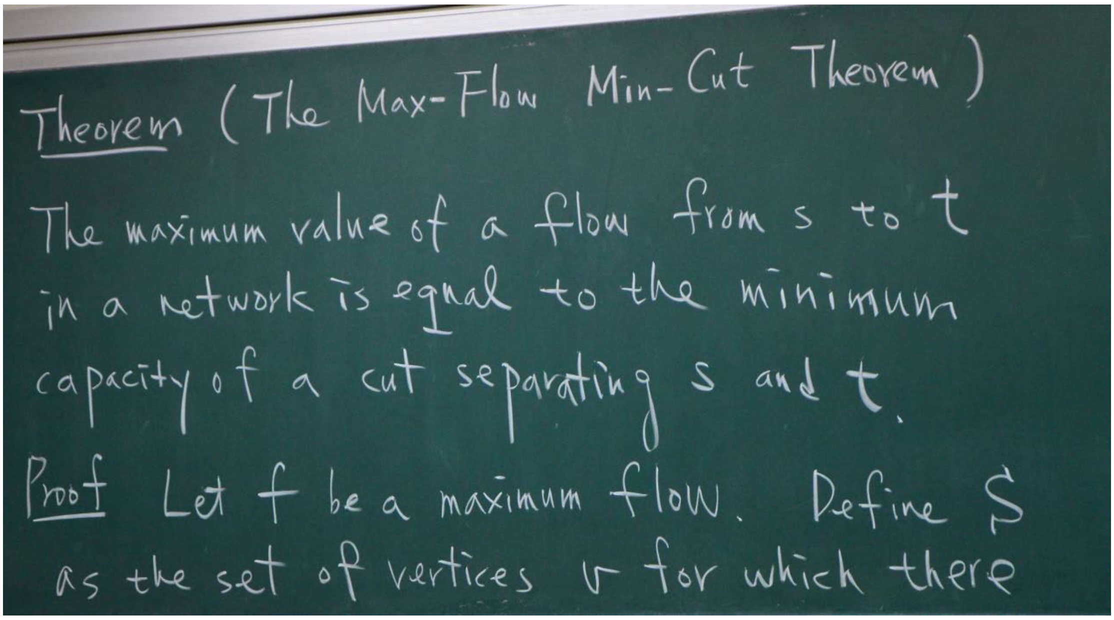
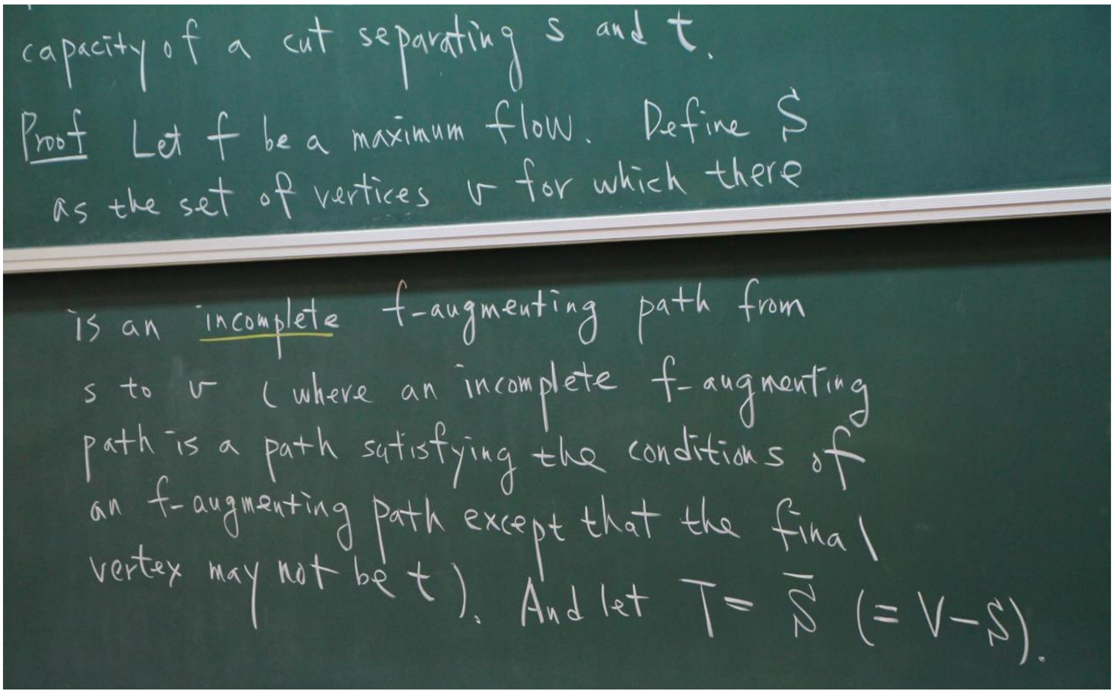
【10920 趙啟超教授離散數學 / 第 32 堂版書】



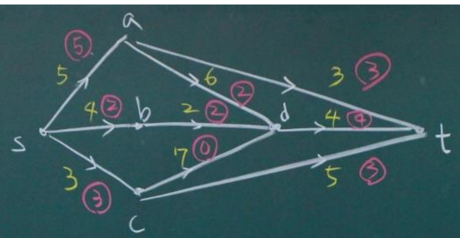


Theorem (The Max-Flow Min-Cut Theorem)

The maximum value of a flow from s to t in a network is equal to the minimum capacity of a cut separating s and t.

Proof Let f be a maximum flow. Define S as the set of vertices v for which there is an incomplete f-augmenting path from s to v (where an incomplete f-augmenting path is a path satisfying the conditions of an f-augmenting path except that the final vertex may not be t). And let $T = \overline{S}$ $(= V - S)$.

(For example,



$S = \{s, b\}$ and $T = \{a, c, d, t\}$)

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The sink t must be in T. Otherwise, we would have an f-augmenting path from s to t and f could be augmented, contradicting the assumption that f is a maximum flow. Hence (S,T) is a cut separating s and t. We then show that $\mathrm{cap}(S,T)=\mathrm{val}(f)$. Let (x,y) be an edge with $x\in S$ and $y\in T$. By definition of S, there is an incomplete

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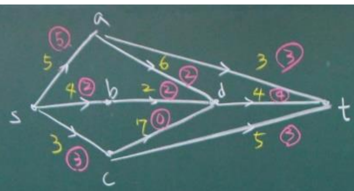

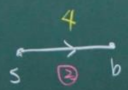

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The sink t must be in T. Otherwise, we would have an f-augmenting path from s to t and f could be augmented, contradicting the assumption that f is a maximum flow. Hence (S,T) is a cut separating s and t. We then show that $\mathrm{cap}(S,T)=\mathrm{val}(f)$. Let (x,y) be an edge with $x\in S$ and $y\in T$. By definition of S, there is an incomplete

f-augmeting path from s to x, and if $f(x,y) < c(x,y)$, we could extend the path to y, contradicting the assumption that y is in T. Thus $f(x,y) = c(x,y)$. Similarly, given an edge (u,v) with $u \in T$ and $v \in S$, there is an imcomplete f-augmenting path from s to v, and if $f(u,v) > 0$, we could extend the path to u,

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contradicting the assumption that u is in T. Thus $f(u,v) = 0$. Therefore,

$$\mathrm{val}(f) = \sum_{\substack{x \in S \\ y \in T}} \underbrace{f(x,y)}_{= c(x,y)} - \sum_{\substack{u \in T \\ v \in S}} \underbrace{f(u,v)}_{= 0}$$

$$= \sum_{\substack{x \in S \\ y \in T}} c(x,y) = \mathrm{cap}(S,T).$$

$\boxed{\text{It follows that } (S,T) \text{ is a minimum cut, as required.}}$

Suppose (S',T') is any other cut. We have $\mathrm{cap}(S',T') \geq \mathrm{val}(f) = \mathrm{cap}(S,T)$.

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Suppose (S',T') is any other cut. We have $\mathrm{cap}(S',T') \geq \mathrm{val}(f) = \mathrm{cap}(S,T)$.

```
procedure Edmonds-Karp
begin
for each e ∈ E
      f(e) := 0
repeat
  use breadth-first search to construct a tree of
  incomplete f-augmenting paths rooted at s
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    if the tree reaches t then
        begin
          augment f
          done := 0
        end
    else done := 1
  until done = 1
  S := the vertex set of the tree
  T := S̄
end
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Remarks

1. When the Edmonds – Karp algorithm ends, the flow f is a maximum flow and (S, T) is a minimum cut.
2. The Edmonds-Karp algorithm is an implementation of the Ford-Fulkerson method using breadth-first search to find f-augmenting paths.